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Note

A class of combinatorial identities ☆

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Abstract

A general theorem for providing a class of combinatorial identities where the sum is over all the partitions of a positive integer is proven. Five examples as the applications of the theorem are given.

Keywords: Combinatorial identities; Partitions; Möbius inversion formula

1. Introduction

If a polynomial has no multiple roots, then it is called separable. For $n \geqslant 1$, let $M_n^{(q)}$ denote the number of monic separable polynomials of degree n over the finite field $\mathbb{F}_q$ with q elements. In 1932, Carlitz [1] showed that $M_1^{(q)} = q$ and $M_n^{(q)} = q^n - q^{n-1}$, for $n > 1$.

The following method for computing the value of $M_n^{(q)}$ is due to Deng [2]. Here we repeat the procedure. Since finite fields are perfect, so any irreducible polynomial over finite fields is separable, and the product of distinct irreducible polynomials is also separable. Let $I_d^{(q)}$ denote the number of monic irreducible polynomials over $\mathbb{F}_q$ of degree d. For a monic separable polynomial over $\mathbb{F}_q$ of degree n, consider its decomposition over $\mathbb{F}_q$ into the product of irreducible factors, we know that each decomposition of the polynomial corresponds to a partition of n. Concretely, suppose

$$\pi(n) = (i_1^{a_1}, \ldots, i_d^{a_d})$$

is a partition of n, where $d > 0$, $i_1 > i_2 > \cdots > i_d > 0$, $a_1 > 0, \ldots, a_d > 0$, and $n = a_1 i_1 + \cdots + a_d i_d$. Let $I_{\pi(n)}^{(q)}$ denote the number of monic separable polynomials over $\mathbb{F}_q$ of degree n which decompose as the product over $\mathbb{F}_q$ of a_j many monic irreducible polynomials of degree i_j, $j = 1, 2, \ldots, d$, then we have

$$I_{\pi(n)}^{(q)} = \prod_{j=1}^{d} \binom{I_{i_j}^{(q)}}{a_j}.$$

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Hence we have

$$M_n^{(q)} = \sum_{\pi(n)} I_{\pi(n)}^{(q)},$$

where the sum is over all the partitions of n. Thus we get the following.

Proposition (*Deng [2]*). *Suppose n is a positive integer. Let $\pi(n)$ denote a partition of n, usually denoted by $(1^{k_1}, 2^{k_2}, \ldots, n^{k_n})$, with $k_1 + 2k_2 + \cdots + nk_n = n$. Then we have that*

$$M_n^{(q)} = \sum_{\pi(n)} \prod_{j=1}^{n} \binom{I_j^{(q)}}{k_j},$$

where the sum is over all the partitions of n.

In view of this proposition and the results of Carlitz, we have the following identity.

$$\sum_{\pi(n)} \prod_{j=1}^{n} \binom{I_j^{(q)}}{k_j} = \begin{cases} q & \text{if } n = 1, \\ q^n - q^{n-1} & \text{if } n > 1, \end{cases}$$

where the sum is over all the partitions of n.

Combinatorial identities have been studied extensively by various authors, e.g. see [3,7] and the references therein. But combinatorial identities where the sum is over all the partitions of a positive integer are rare. Motivating by the above identity, in this paper we will consider a class of combinatorial identities where the left-hand side is a sum over all the partitions of a positive integer. We get a class of such identities. It is remarkable that these identities have a common profile, that is it involves a parameter in an infinite set.

2. Main theorem and the proof

First let us fix some notations. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive integers and let $\mathbb{R}$ denote the set of all reals. Let μ denote the Möbius function and let log denote the natural logarithm. Define binomial coefficients as follows:

$$\binom{x}{0} = 1, \quad \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \quad \forall n \in \mathbb{N}, \ x \in \mathbb{R}.$$

Theorem. *Suppose Λ is a nonempty subset of $\mathbb{R}$ (called space of parameters). For each $\lambda \in \Lambda$, let $a_1^{(\lambda)}, a_2^{(\lambda)}, \ldots$ be a sequence of reals. The sequence $(b_n^{(\lambda)})$ is defined by*

$$b_n^{(\lambda)} = \sum_{d \mid n} d a_d^{(\lambda)}.$$

Assume that the sequence $(b_n^{(\lambda)})$ satisfies one of the following two conditions:

(i) *there exists a function $f : \Lambda \times \mathbb{N} \longrightarrow \Lambda$ such that*

$$b_{nm}^{(\lambda)} = b_m^{(f(\lambda,n))} \quad \forall n, m \in \mathbb{N}, \lambda \in \Lambda.$$

(ii)

$$b_{nm}^{(\lambda)} = b_n^{(\lambda)} + b_m^{(\lambda)} \quad \forall n, m \in \mathbb{N}, \lambda \in \Lambda.$$

The sequence $(c_n^{(\lambda)})$ *is defined by*

$$c_n^{(\lambda)} = \begin{cases} \dfrac{b_n^{(\lambda)}}{n} & \text{if } n \text{ is odd}, \\ \dfrac{b_n^{(\lambda)}}{n} - \dfrac{b_{n/2}^{(\lambda)}}{n/2} & \text{if } n \text{ is even}. \end{cases}$$

Then we have the following identity

$$\sum_{\pi(n)} \prod_{j=1}^{n} \binom{a_j^{(\lambda)}}{k_j} = \sum_{i=1}^{n} \frac{1}{i!} \sum_{j_1+\ldots+j_i=n} c_{j_1}^{(\lambda)} \cdots c_{j_i}^{(\lambda)} \quad \forall n \in \mathbb{N}, \lambda \in \Lambda,$$

where the sum of the left-hand side is over all the partitions of n.

Remark. The theorem in essence transforms a sum over partitions into another sum. In general, sums over partitions are difficult to handle (see p. 224 of [3] as an open problem 4.2). Using this general theorem, in the next section we can transform several sums over partitions into ordinary sums.

To prove this theorem we need a lemma.

Lemma. *Using the notations as in the theorem. For* $n \geqslant 1$, *it holds that*

$$\frac{1}{n} \sum_{d \mid n} (-1)^{n/d-1} d a_d^{(\lambda)} = \begin{cases} \dfrac{b_n^{(\lambda)}}{n} & \text{if } n \text{ is odd}, \\ \dfrac{b_n^{(\lambda)}}{n} - \dfrac{b_{n/2}^{(\lambda)}}{n/2} & \text{if } n \text{ is even}. \end{cases}$$

Proof. If n is odd, then the result follows from the definition of $b_n^{(\lambda)}$.

Suppose now n is even. We write n as $n = 2^\alpha \beta$, where $\alpha > 0$ and β is odd. Any divisor d of n can be written as $d = 2^i j$, where $0 \leqslant i \leqslant \alpha$ and $j \mid \beta$. Thus we have that

$$\frac{1}{n} \sum_{d \mid n} (-1)^{n/d-1} d a_d^{(\lambda)} = \frac{1}{n} \left(\sum_{j \mid \beta} 2^\alpha j a_{2^\alpha j}^{(\lambda)} - \sum_{\substack{0 \leqslant i < \alpha \\ j \mid \beta}} 2^i j a_{2^i j}^{(\lambda)} \right).$$

Obviously, by the definition of $b_n^{(\lambda)}$ we have that

$$\sum_{\substack{0 \leqslant i < \alpha \\ j \mid \beta}} 2^i j a_{2^i j}^{(\lambda)} = \sum_{d \mid 2^{\alpha-1}\beta} d a_d^{(\lambda)} = \sum_{d \mid \frac{n}{2}} d a_d^{(\lambda)} = b_{n/2}^{(\lambda)}.$$

From the Möbius inversion formula and the properties of Möbius function, we have

$$2^\alpha j a_{2^\alpha j}^{(\lambda)} = \sum_{d \mid 2^\alpha j} \mu(d) b_{2^\alpha j/d}^{(\lambda)} = \sum_{d \mid j} \mu(d) b_{2^\alpha j/d}^{(\lambda)} - \sum_{d \mid j} \mu(d) b_{2^{\alpha-1} j/d}^{(\lambda)}.$$

Assume that the sequence $(b_n^{(\lambda)})$ satisfies Condition (i). Then

$$\begin{aligned} &\sum_{d \mid j} \mu(d) b_{2^\alpha j/d}^{(\lambda)} - \sum_{d \mid j} \mu(d) b_{2^{\alpha-1} j/d}^{(\lambda)} \\ &\quad = \sum_{d \mid j} \mu(d) b_{j/d}^{(f(\lambda,2^\alpha))} - \sum_{d \mid j} \mu(d) b_{j/d}^{(f(\lambda,2^{\alpha-1}))} \\ &\quad = j a_j^{(f(\lambda,2^\alpha))} - j a_j^{(f(\lambda,2^{\alpha-1}))}. \end{aligned}$$

Hence

$$\sum_{j\,|\,\beta} 2^\alpha j a^{(\lambda)}_{2^\alpha j} = \sum_{j\,|\,\beta}\left(j a_j^{(f(\lambda,2^\alpha))} - j a_j^{(f(\lambda,2^{\alpha-1}))}\right)$$
$$= b_\beta^{(f(\lambda,2^\alpha))} - b_\beta^{(f(\lambda,2^{\alpha-1}))} = b^{(\lambda)}_{2^\alpha\beta} - b^{(\lambda)}_{2^{\alpha-1}\beta} = b_n^{(\lambda)} - b_{n/2}^{(\lambda)}.$$

Therefore

$$\frac{1}{n}\sum_{d\,|\,n}(-1)^{n/d-1} d a_d^{(\lambda)} = \frac{1}{n}(b_n^{(\lambda)} - b_{n/2}^{(\lambda)} - b_{n/2}^{(\lambda)}) = \frac{b_n^{(\lambda)}}{n} - \frac{b_{n/2}^{(\lambda)}}{n/2}.$$

If the sequence $\left(b_n^{(\lambda)}\right)$ satisfies Condition (ii), similar to the above argument, we can prove the same result. This completes the proof of the lemma. □

Proof of the theorem. The following arguments work for sufficiently small real variable x. Consider the function defined by

$$g(x) = \prod_{j=1}^{\infty}(1+x^j)^{a_j^{(\lambda)}}.$$

Its power series expansion is

$$g(x) = \prod_{j=1}^{\infty}\left(\sum_{k_j=0}^{\infty}\binom{a_j^{(\lambda)}}{k_j} x^{jk_j}\right)$$
$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{\pi(n)}\prod_{j=1}^{n}\binom{a_j^{(\lambda)}}{k_j}\right)x^n.$$

On the other hand, by the lemma we have

$$\log g(x) = \sum_{j=1}^{\infty} a_j^{(\lambda)}\log(1+x^j) = \sum_{j=1}^{\infty} a_j^{(\lambda)} \sum_{l=1}^{\infty}(-1)^{l-1}\frac{x^{jl}}{l}$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{d\,|\,n}(-1)^{\frac{n}{d}-1} d a_d^{(\lambda)}\right)x^n$$
$$= \sum_{n=1}^{\infty} c_n^{(\lambda)} x^n.$$

Hence

$$g(x) = \exp\left(\sum_{n=1}^{\infty} c_n^{(\lambda)} x^n\right)$$
$$= 1 + \sum_{j=1}^{\infty}\frac{1}{j!}\left(\sum_{n=1}^{\infty} c_n^{(\lambda)} x^n\right)^j$$
$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{1}{i!}\sum_{j_1+\cdots+j_i=n} c_{j_1}^{(\lambda)}\cdots c_{j_i}^{(\lambda)}\right)x^n.$$

Therefore

$$\sum_{\pi(n)} \prod_{j=1}^{n} \binom{a_j^{(\lambda)}}{k_j} = \sum_{i=1}^{n} \frac{1}{i!} \sum_{j_1+\cdots+j_i=n} c_{j_1}^{(\lambda)} \cdots c_{j_i}^{(\lambda)}.$$

This completes the proof of the theorem. □

3. Examples

In this section, we will get some concrete identities by means of the theorem.

Example 1. As a first example, we see how the identity at the beginning of the paper falls into our theorem. Let Λ be the set of all prime powers of $\mathbb{N}$. For $q \in \Lambda$, $a_n^{(q)} = I_n^{(q)}$. From [6], $b_n^{(q)} = q^n$. And $f(q, n) = q^n$. It is easy to see that the condition $b_{nm}^{(q)} = b_m^{(f(q,n))}$ holds.

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{(q)} x^n &= \sum_{n=1}^{\infty} \frac{q^n}{n} x^n - \sum_{n=1}^{\infty} \frac{q^n}{n} x^{2n} \\ &= \log \frac{1-qx^2}{1-qx}.\end{aligned}$$

Thus

$$g(x) = \frac{1-qx^2}{1-qx} = 1 + qx + \sum_{n=2}^{\infty} (q^n - q^{n-1}) x^n.$$

So we get the preceding identity by the theorem.

Example 2. This is an example from combinatorial theory (see p. 12 of [4]). Let Λ be the set of all integers greater than 1. For $r \in \Lambda$, let $M_n^{(r)}$ denote the number of circular sequences of length and period n over the set $\{1, 2, \ldots, r\}$. Then

$$\sum_{d \mid n} \mathrm{d} M_d^{(r)} = r^n,$$

$$M_n^{(r)} = \frac{1}{n} \sum_{d \mid n} \mu(d) r^{n/d}.$$

Completely similar to Example 1 we have the following identity.

$$\sum_{\pi(n)} \prod_{j=1}^{n} \binom{M_j^{(r)}}{k_j} = \begin{cases} r & \text{if } n = 1, \\ r^n - r^{n-1} & \text{if } n > 1. \end{cases}$$

The following three examples need some elementary facts from number theory, one can see Hardy and Wright [5].

Example 3. It is well-known that

$$\sum_{d \mid n} \mu(d) = \delta_{1,n},$$

where $\delta_{1,n}$ is the Kronecker symbol. We put $\Lambda = \mathbb{R}$ and $a_n^{(\lambda)} = \lambda \mu(n)/n$. So $b_n^{(\lambda)} = \lambda \delta_{1,n}$. Define $f(\lambda, n) = \lambda \delta_{1,n}$. Here

$$\sum_{n=1}^{\infty} c_n^{(\lambda)} x^n = \lambda x - \lambda x^2.$$

It is easy to see that

$$\exp(\lambda x-\lambda x^2)=1+\sum_{n=1}^{\infty}\left(\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{\lambda^{n-i}}{(n-i)!}\binom{n-i}{i}\right)x^n.$$

Thus we have the following identity

$$\sum_{\pi(n)}\prod_{j=1}^{n}\binom{\lambda\mu(j)/j}{k_j}=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{\lambda^{n-i}}{(n-i)!}\binom{n-i}{i}.$$

Example 4. Let ϕ denote Euler phi-function. It is well-known that

$$\sum_{d\mid n}\phi(d)=n.$$

We put $\Lambda=\mathbb{R}$ and $a_n^{(\lambda)}=\lambda\phi(n)/n$. So $b_n^{(\lambda)}=\lambda n$. Define $f(\lambda,n)=\lambda n$. Here

$$\sum_{n=1}^{\infty}c_n^{(\lambda)}x^n=\frac{\lambda x}{1-x^2}.$$

It is easy to see that

$$\exp\left(\frac{\lambda x}{1-x^2}\right)=1+\sum_{n=1}^{\infty}\left(\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\frac{\lambda^{n-2i}}{(n-2i)!}\binom{n-i-1}{i}\right)x^n.$$

Thus we have the following identity

$$\sum_{\pi(n)}\prod_{j=1}^{n}\binom{\lambda\phi(j)/j}{k_j}=\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\frac{\lambda^{n-2i}}{(n-2i)!}\binom{n-i-1}{i}.$$

Example 5. In the above four examples, the sequence $(b_n^{(\lambda)})$ satisfies Condition (i). In this example the sequence $(b_n^{(\lambda)})$ satisfies Condition (ii). Consider von Mangoldt function $\Lambda(n)$. It is defined by

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^m,\ m\geqslant 1,\ p \text{ is a prime},\\ 0 & \text{otherwise.}\end{cases}$$

It is well-known that

$$\sum_{d\mid n}\Lambda(d)=\log n.$$

We put the space of parameters be $\mathbb{R}$ and $a_n^{(\lambda)}=\lambda\Lambda(n)/n$. So $b_n^{(\lambda)}=\lambda\log n$ and it satisfies Condition (ii). By the theorem we can get the corresponding identity. Here we omit the details.

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